August 1998

1 Day 1, Question 1

Establish the radius of convergence for the power series expansion about $x = 0$ of

$$f(x) = \frac{\sin^2(x)}{1 + x^{51}}.$$

When viewed as a function of a complex variable z, f is meromorphic, with a simple pole at each of the 51st roots of unity and no other poles. This ensures that in some open disc in the complex plane, centered at $z = 0$, the power series for $f(z)$ converges uniformly to $f(z)$. Since $f(z)$ does not exist at each of the 51st roots of unity, the power series cannot converge at any of these roots. Each of these roots has modulus 1, so the power series has radius of convergence 1.

2 Day 1, Question 3

Consider the linear system

$$\begin{pmatrix} \beta & 1 & \beta \\ 0 & 3 & 0 \\ 1 & 0 & \beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

How many solutions does this system have as a function of the real parameter β?

Since this is a linear system, for a fixed value of β there may be no solution, one solution, or an infinite number of solutions.

Consider the augmented matrix $(\boldsymbol{A}|\boldsymbol{b})$.

$$\left(\begin{array}{ccc|c} \beta & 1 & \beta & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & \beta & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \beta & 1 \\ 0 & 3 & 0 & 0 \\ \beta & 1 & \beta & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \beta & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & \beta - \beta^2 & -\beta \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \beta & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & \beta - \beta^2 & -\beta \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \beta & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta - \beta^2 & -\beta \end{array}\right)$$

If $\beta = 1$, then the last line would require $0 \cdot z = -1$. Hence if $\beta = 1$, there are no solutions. On the other hand, if $\beta \neq 1$ and $\beta \neq 0$, then the left part of the augmented matrix is upper triangular with nonzero elements on the diagonal, so it is invertible. This means that if $\beta \notin \{0, 1\}$, then there is one solution. What happens if $\beta = 0$?

$$\left(\begin{array}{ccc|c} 1 & 0 & \beta & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta - \beta^2 & -\beta \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The augmented matrix indicates that $x = 1$, $y = 0$, and there are no restrictions on z. z is a free variable, so there is an infinite number of solutions, all of the form

$$\begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix},$$

where z can be any real number.

3 Day 1, Question 4

Evaluate the line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

where

$$\boldsymbol{F} = (x + 2y + 4z)\ \boldsymbol{i} + (2x - 3y - z)\ \boldsymbol{j} + (4x - y + 2z)\ \boldsymbol{k}$$

and the curve C is given parametrically by

$$(x(t), y(t), z(t)) = (t^2, t^3, \sin(\pi t)), \quad t \in [0, 2].$$

First we test whether the vector field $\boldsymbol{F}$ is conservative.

$$\begin{aligned} \nabla \times \boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + 2y + 4z & 2x - 3y - z & 4x - y + 2z \end{vmatrix} \\ &= (-1 - (-1))\ \boldsymbol{i} + (4 - 4)\ \boldsymbol{j} + (2 - 2)\ \boldsymbol{k} \ = \ \boldsymbol{0} \end{aligned}$$

That $\boldsymbol{F}$ has zero curl indicates that $\boldsymbol{F} = \nabla\phi$ for some scalar function ϕ. The contour integral satisfies

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \nabla\phi \cdot \dot{\boldsymbol{r}} = \phi(\boldsymbol{r}_{\text{end}}) - \phi(\boldsymbol{r}_{\text{start}}),$$

where $\boldsymbol{r}_{\text{start}}$ and $\boldsymbol{r}_{\text{end}}$ are the start and end points of the curve, respectively. We must find the function ϕ.

Since $\nabla\phi = \boldsymbol{F}$,

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= x + 2y + 4z, \\ \frac{\partial\phi}{\partial y} &= 2x - 3y - z, \\ \frac{\partial\phi}{\partial z} &= 4x - y + 2z. \end{aligned}$$

The first of these tells us

$$\phi(x,y,z) = \frac{1}{2}x^2 + 2xy + 4xz + f(y,z).$$

The partial derivative of this with respect to y is

$$\phi_y(x,y,z) = 2x + f_y(y,z).$$

Combined with the known form of the partial derivative ϕ_y, this indicates that

$$f_y(y,z) = -3y - z.$$

This, in turn, implies

$$f(y,z) = -\frac{3}{2}y^2 - yz + g(z).$$

So far we have

$$\begin{aligned} \phi(x,y,z) &= \frac{1}{2}x^2 + 2xy + 4xz - \frac{3}{2}y^2 - yz + g(z), \\ \text{so that} \quad \phi_z(x,y,z) &= 4x - y + g'(z). \end{aligned}$$

Referring the earlier form of ϕ_z, we conclude that $g'(z) = 2z$, so

$$\begin{aligned} g(z) &= z^2 + c, \\ \text{so} \quad \phi(x,y,z) &= \frac{1}{2}x^2 + 2xy + 4xz - \frac{3}{2}y^2 - yz + z^2 + c, \end{aligned}$$

where c is a constant. When we check our work and compute the partial derivatives of this ϕ, we reproduce the components of $\boldsymbol{F}$.

The start and end points of the curve C are

$$(x(0),y(0),z(0)) = (0,0,0), \qquad \text{and} \qquad (x(2),y(2),z(2)) = (4,8,0),$$

respectively. The value of the line integral is then

$$\begin{aligned} \phi(4,8,0) - \phi(0,0,0) &= \left(\frac{1}{2}\cdot 4^2 + 2\cdot 4\cdot 8 + 4\cdot 4\cdot 0 - \frac{3}{2}\cdot 8^2 - 8\cdot 0 + 0^2 + c\right) - c \\ &= -32. \end{aligned}$$

4 Day 1, Question 5

Given that

$$\lim_{s\to 0}(1+s)^{1/s} = e,$$

evaluate

$$\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx.$$

Justify the steps in your computation.

First note that for fixed $x \geq 0$,

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^{n/x}\right)^x \\
&= \left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n/x}\right)^x \quad \text{by continuity of } \; r \mapsto r^x \\
&= \left(\lim_{x/n\to 0}\left(1+\frac{x}{n}\right)^{(x/n)^{-1}}\right)^x \\
&= (e)^x \; = \; e^x.
\end{aligned}$$

This looks like a job for Lebesgue's Dominated Convergence Theorem or Lebesgue's Monotone Convergence Theorem. To use the former, we must establish that there is some integrable function that dominates each integrand for almost all x. To use the latter, we must show that the integrands form a monotone (in n) sequence. Your suspicion should be that the integrands are monotone increasing in n and that the limit function is $e^x e^{-2x} = e^{-x}$, which is integrable on $[0, \infty)$.

Recall the Taylor series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots ,$$

which has infinite radius of convergence. Since each coefficient is positive, for $x \geq 0$ we have

$$1 + \frac{x}{n} \;\leq\; e^{x/n}, \quad \text{so} \quad \left(1+\frac{x}{n}\right)^n \;\leq\; \left(e^{x/n}\right)^n \;=\; e^x.$$

Hence each integrand is dominated by $e^x e^{-2x} = e^{-x}$, which is integrable on $[0, \infty)$. By Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx &= \int_0^\infty \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n e^{-2x}\,dx \\
&= \int_0^\infty e^x e^{-2x}\,dx \\
&= \int_0^\infty e^{-x}\,dx \;=\; 1.
\end{aligned}$$

5 Day 1, Question 6

Set up a numerical scheme to solve the following system on the unit square $\Gamma = \{(x, y) : 0 \leq x, y \leq 1\}$

$$\begin{aligned} -u_{xx} - u_{yy} + u &= 0 \text{ on } \Gamma, \\ u(x, y) &= f(x, y) \text{ on } \partial\Gamma, \end{aligned}$$

so that

1. the maximum principle is satisfied
2. the resulting linears system can be solved.

You should show that these two properties hold.

6 Day 1, Question 7

Determine if the following integral converges.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(xy+yz+zx)} \, dx \, dy \, dz$$

If it converges, finds its value.

7 Day 1, Question 8

Consider the ODE

$$\frac{dA}{dt} = \mu A - |A|^2 A,$$

where A is complex and μ is real. Find the fixed points of this dynamical system as a function of μ and perform their linear stability analysis.

8 Day 1, Question 9

For each $\theta \in [0, 1]$, consider the numerical scheme

$$y_{n+1} = y_n + hf(t_n + \theta h, \theta y_{n+1} + (1 - \theta)y_n)$$

to solve $y' = f(t, y)$. Find the order of its truncation error as a function of θ. Assume that f is smooth.

If we use the definition of local truncation error found in *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations* by Ascher and Petzold, then we consider the size of the difference

$$\frac{y(t_n+h)-y(t_n)}{h} - hf\left(t_n+\theta h, \theta y(t_n+h)+(1-\theta)y(t_n)\right),$$

where y is the exact solution of the differential equation.

Note that for $\theta = 0, 1/2, 1$ we have well-known methods.

θ	Scheme	Name	Order of loc. trunc. error
0	$y_{n+1} = y_n + hf(t_n, y_n)$	forward Euler method	h
$\frac{1}{2}$	$y_{n+1} = y_n + hf\left(t_n + \frac{1}{2}h, \frac{1}{2}y_{n+1} + \frac{1}{2}y_n\right)$	midpoint method	h^2
1	$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$	backward Euler method	h

The fourth basic method that should be memorized is the trapezoidal method, which has local truncation error of order h^2:

$$y_{n+1} = y_n + \frac{1}{2}h\left(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)\right).$$

The Taylor expansion of f will be simpler if we make $(t_n+\theta h, y(t_n+\theta h))$ the "center" of the expansion. Keeping track of terms will be easier if we adopt the following shorthand:

$$\begin{aligned}
t_{n+\theta} &:= t_n + \theta h, \\
y_{n+\theta} &:= y(t_n+\theta h), \\
y'_{n+\theta} &:= y'(t_n+\theta h) \quad = \quad f(t_n+\theta h, y(t_n+\theta h)), \\
y''_{n+\theta} &:= y''(t_n+\theta h), \\
y^{(3)}_{n+\theta} &:= y^{(3)}(t_n+\theta h).
\end{aligned}$$

First we consider $y_n = y(t_n)$ and $y_{n+1} = y(t_{n+1})$.

$$\begin{aligned}
y_{n+1} &= y_{n+\theta} + y'_{n+\theta}(1-\theta)h + \frac{1}{2}y''_{n+\theta}(1-\theta)^2h^2 + \mathcal{O}\left(h^3\right) \\
y_n &= y_{n+heta} + y'_{n+\theta}(-\theta h) + \frac{1}{2}y''_{n+\theta}(-\theta h)^2 + \mathcal{O}\left(h^3\right) \\
y_{n+1} - y_n &= y'_{n+\theta}(1-\theta-(-\theta))h + \frac{1}{2}y''_{n+\theta}((1-\theta)^2-\theta^2)h^2 + \mathcal{O}\left(h^3\right) \\
&= y'_{n+\theta}h + \frac{1}{2}y''_{n+\theta}(1-2\theta)h^2 + \mathcal{O}\left(h^3\right)
\end{aligned}$$

Since we will expand f in its y argument, we should consider the difference between $y_{n+\theta}$ and $\theta y_{n+1} + (1-\theta)y_n$.

$$\begin{aligned}\theta y_{n+1} + (1-\theta)y_n - y_{n+\theta} &= \theta\left(y_{n+\theta} + y'_{n+\theta}(1-\theta)h + \frac{1}{2}y''_{n+\theta}(1-\theta)^2h^2\right) \\ &+ (1-\theta)\left(y_{n+\theta} + y'_{n+\theta}(-\theta h) + \frac{1}{2}y''_{n+\theta}(-\theta h)^2\right) - y_{n+\theta} + \mathcal{O}\left(h^3\right) \\ &= \frac{1}{2}y''_{n+\theta}\left(\theta(1-\theta)^2 + (1-\theta)\theta^2\right)h^2 + \mathcal{O}\left(h^3\right) \\ &= \frac{1}{2}y''_{n+\theta}\left(\theta-\theta^2\right)h^2 + \mathcal{O}\left(h^3\right)\end{aligned}$$

We use this in the expansion of f in its y argument.

$$\begin{aligned}f\left(t_{n+\theta}, \theta y_{n+1} + (1-\theta)y_n\right) &= f\left(t_{n+\theta}, y_{n+\theta}\right) + f_y\left(t_{n+\theta}, y_{n+\theta}\right)\frac{1}{2}y''_{n+\theta}\left(\theta-\theta^2\right)h^2 \\ &+ \frac{1}{2}f_{yy}\left(t_{n+\theta}, y_{n+\theta}\right)\left(\frac{1}{2}y''_{n+\theta}\left(\theta-\theta^2\right)h^2\right)^2 + \mathcal{O}\left(h^6\right) \\ &= y'_{n+\theta} + f_y\left(t_{n+\theta}, y_{n+\theta}\right)\frac{1}{2}y''_{n+\theta}\left(\theta-\theta^2\right)h^2 + \mathcal{O}\left(h^4\right)\end{aligned}$$

Finally, we compute the local truncation error.

$$\begin{aligned}&\left(\frac{y_{n+1}-y_n}{h}\right) - f\left(t_{n+\theta}, \theta y_{n+1} + (1-\theta)y_n\right) \\ &= \left(y'_{n+\theta} + \frac{1}{2}y''_{n+\theta}(1-2\theta)h + \mathcal{O}\left(h^2\right)\right) - \left(y'_{n+\theta} + f_y\left(t_{n+\theta}, y_{n+\theta}\right)\frac{1}{2}y''_{n+\theta}\left(\theta-\theta^2\right)h^2 + \mathcal{O}\left(h^4\right)\right) \\ &= \frac{1}{2}y''_{n+\theta}(1-2\theta)h - f_y\left(t_{n+\theta}, y_{n+\theta}\right)\frac{1}{2}y''_{n+\theta}\left(\theta-\theta^2\right)h^2 + \mathcal{O}\left(h^2\right)\end{aligned}$$

If $\theta = 1/2$, then the first term in the last line vanishes. All that remains is of order h^2 and higher, so if $\theta = 1/2$, then the local truncation error is of order h^2. If $\theta \neq 1/2$, then the first line does not vanish, and the term of order h remains. We conclude that if $\theta \neq 1/2$, the local truncation error is of order h.

9 Day 1, Question 10

The Schrödinger equation of the hydrogen atom is

$$-\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{x}) - \frac{e^2}{|\boldsymbol{x}|}\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x}).$$

where $\hbar$, m, and e are physical constants, E is the energy level of the atom, and $\psi(\boldsymbol{x})$ is a function on $\mathbb{R}^3$. (Δ is the three-dimensional Laplacian.) The dimensionless form of this equation is

$$\left(-\Delta - \frac{2}{|\boldsymbol{y}|}\right)\phi(\boldsymbol{y}) = \lambda\phi(\boldsymbol{y}).$$

where $\phi(\boldsymbol{y})$ is a function on $\mathbb{R}^3$. The function $\phi(\boldsymbol{y}) = e^{-|\boldsymbol{y}|}$ solves this equation with $\lambda = -1$. Find the energy level E of the original equation corresponding to this soltuion of the dimensionless equation. Your answer should be in terms of e, $\hbar$, and m. Note the units

$$\begin{aligned} \left[\frac{\hbar^2}{2m}\right] &= \text{energy} \times \text{length}^2, \\ \left[e^2\right] &= \text{energy} \times \text{length}, \\ [E] &= \text{energy}. \end{aligned}$$

10 Day 1, Question 11

Let $C([0,1])$ denote the space of real-valued, continuous functions over the interval $[0,1]$. This space is equipped with the usual sup norm $\|f\|_\infty := \sup_{x\in[0,1]}|f(x)|$. Let E be the set of functions f in $C([0,1])$ such that $|f(x)| \le x$, over $0 \le x \le 1$.

Prove or disprove: E is compact.

The given set E is not compact. In a compact metric space, each infinite sequence has a convergent subsequence. We provide an infinite sequence in E that does not have a convergent subsequence.

Define the functions $f_n(x) = x^n$, $n \in \mathbb{N}$, and note that $f_n \in C([0,1])$ and $|f_n(x)| = x^n \le x$ for $x \in [0,1]$. $\{f_n\}$ is an infinite sequence in E.

Suppose that there is some $f \in E$ such that $f_n \to f$ with respect to the sup-norm. Then $f_n \to f$ pointwise as well. But we know that f_n converges pointwise to the discontinuous function

$$g(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

Further, since $\{f_n\}$ is a monotone sequence ($f_n(x) \leq f_m(x)$ for each $x \in [0,1]$ if $m \leq x$), each subsequence converges pointwise to the same function $g \notin E$.

Since E contains an infinite sequence that has no subsequence that converges in sup-norm to some function in E, E is not compact.

11 Day 1, Question 12

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, connected set with a smooth boundary. Consider the biharmonic operator $\Delta^2 = \Delta\Delta$ acting on the domain

$$\mathcal{D}\left(\Delta^2\right) = \left\{f \in C^4\left(\overline{\Omega}\right) : f = 0,\ \boldsymbol{n} \cdot \nabla f = 0 \ \text{ on } \partial\Omega\right\},$$

where $\boldsymbol{n}$ is the unit vector normal to the boundary. Prove that this operator is non-negative definite with respect to the L^2 inner product.

Recall first what it means for an $n \times n$ matrix $\boldsymbol{A}$ to be non-negative definite. $\boldsymbol{A}$ is non-negative definite if for each $\boldsymbol{x} \in \mathbb{C}^n$,

$$(\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}) \ = \ \overline{\boldsymbol{x}}^T \boldsymbol{A}\boldsymbol{x} \ \geq \ 0.$$

There are subtleties in the case of a linear differential operator that cannot be captured by a finite-dimensional analogue such as this, but the similarities are significant. Just as $\boldsymbol{A}\boldsymbol{x}$ is a vector in $\mathbb{C}^n$ and has dot-products with all other vectors in $\mathbb{C}^n$, Lf is a vector in $L^2(\Omega)$ and has inner products with all other vectors in $L^2(\Omega)$. If the inner product of Lf with f is non-negative for all $f \in \mathcal{D}(L)$, then L is said to be non-negative definite.

Your intuition should be that, when taking the L^2 inner product

$$\left(f, \Delta^2 f\right)_{L^2} = \int_\Omega f \ \Delta^2 f \ dv,$$

there should be a way to move one of the Laplacians over to the "bare" f. This would solve the problem, for then we'd have

$$\int_\Omega \Delta f \ \Delta f \ dv \ = \ \int_\Omega |\Delta f|^2 \ dv \ \geq \ 0,$$

as the integral of the absolute value of some function must be non-negative.

The conditions given for the domain $\mathcal{D}\left(\Delta^2\right)$ will ensure the vanishing of "surface terms" arising from the application of the Divergence Theorem.

For simplicity of notation, let $g = \Delta f$. Then $\Delta^2 f = \Delta g$, and

$$\left(f, \Delta^2 f\right)_{L^2} \ = \ (f, \Delta g)_{L^2} \ = \ \int_\Omega f \ \Delta g \ dv.$$

Note that $\Delta g = \nabla \cdot \nabla g$. The need to employ a vectorial version of integration-by-parts inspires the following vectorial version of Leibniz's Rule:

$$\nabla \cdot (f \nabla g) \;=\; (\nabla f) \cdot (\nabla g) + f\,(\nabla \cdot \nabla g) \;=\; (\nabla f) \cdot (\nabla g) + f \Delta g.$$

Hence

$$f \Delta g \;=\; \nabla \cdot (f \nabla g) - (\nabla f) \cdot (\nabla g)\,,$$

and

$$\begin{aligned} \int_\Omega f\ \Delta g\ dv \;&=\; \int_\Omega [\nabla \cdot (f \nabla g) - (\nabla f) \cdot (\nabla g)]\, dv \\ &=\; \int_\Omega \nabla \cdot (f \nabla g)\ dv - \int_\Omega (\nabla f) \cdot (\nabla g)\ dv. \end{aligned}$$

If we apply the Divergence Theorem to the first integral, we find

$$\int_\Omega \nabla \cdot (f\ \nabla g)\ dv \;=\; \oint_{\partial\Omega} (f\ \nabla g) \cdot \boldsymbol{n}\ da \;=\; 0$$

because $f = 0$ on $\partial\Omega$.

So far we have

$$\int_\Omega f\ \Delta g\ dv \;=\; -\int_\Omega (\nabla f) \cdot (\nabla g)\ dv.$$

We have successfully "transferred" one ∇ from g to f. Now we seek to transfer the remaining ∇. Remember that we want to have Δf times itself in the final expression, and remember that $g = \Delta f$. We appeal to Leibniz's Rule again:

$$\nabla \cdot (g\ \nabla f) \;=\; (\nabla g) \cdot (\nabla f) + g\ \nabla \cdot \nabla f \;=\; (\nabla g) \cdot (\nabla f) + g\ \Delta f,$$

so

$$(\nabla g) \cdot (\nabla f) \;=\; \nabla \cdot (g\ \nabla f) - g\ \Delta f,$$

and

$$\begin{aligned} -\int_\Omega (\nabla f) \cdot (\nabla g)\ dv \;&=\; -\int_\Omega [\nabla \cdot (g\ \nabla f) - g\ \Delta f]\, dv \\ &=\; \int_\Omega g\ \Delta f\ dv - \int_\Omega \nabla \cdot (g\ \nabla f)\ dv. \end{aligned}$$

We apply the Divergence Theorem to the second integral and find

$$\int_\Omega \nabla \cdot (g\ \nabla f)\ dv \;=\; \oint_{\partial\Omega} (g\ \nabla f) \cdot \boldsymbol{n}\ da \;=\; 0$$

because $\boldsymbol{n} \cdot \nabla f = 0$ on $\partial\Omega$.

We have shown that

$$\int_\Omega f\ \Delta g\ dv = \int_\Omega g\ \Delta f\ dv.$$

Since $g = \Delta f$, we have

$$\int_\Omega f\ \Delta g\ dv \;=\; \int_\Omega \Delta f\ \Delta f\ dv \;=\; \int_\Omega |\Delta f|^2\ dv \;\geq\; 0.$$

12 Day 2, Question 1

Let f_n, $n \in \mathbb{Z}$, be complex-valued functions on $\mathbb{R}$ defined by

$$f_n(x) = \frac{1}{\sqrt{\pi}} \frac{(x-i)^n}{(x+i)^{n+1}}.$$

Prove that these functions are an orthonormal set with respect to the inner product

$$(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

(Hint: Consider using contour integration.)

First we demonstrate the normalization.

$$\begin{aligned}
(f_n, f_n) &= \int_{-\infty}^{\infty} f(x)\overline{f(x)}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{(x-i)^n}{(x+i)^{n+1}} \cdot \frac{(x+i)^n}{(x-i)^{n+1}}\, dx \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x^2+1)^n}{(x^2+1)^{n+1}}\, dx \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx \\
&= \frac{1}{\pi} \lim_{R\to\infty} \arctan(x)\Big|_{x=-R}^{x=R} \\
&= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\
&= 1
\end{aligned}$$

Now suppose $m = n + k + 1$, where $k \geq 0$.

$$\begin{aligned}
(f_n, f_m) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-i)^n}{(x+i)^{n+1}} \cdot \frac{(x+i)^m}{(x-i)^{m+1}}\, dx \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x+i)^k}{(x-i)^{k+2}}\, dx
\end{aligned}$$

We now consider the integrand as a function of a complex argument $z = x+iy$, and we consider the integral along the contour Γ consisting of these two pieces:

$$\begin{aligned}
z &= Re^{i\theta}, \quad 0 \leq \theta \leq \pi, \\
z &= x, \quad -R \leq x \leq R.
\end{aligned}$$

Inside this contour the integrand is meromorphic (analytic expected for poles). It has a pole of order $k+2$ at $z = i$. If $k = 0$, then the pole at $z = i$ is of order

2, and the residue is

$$\frac{d}{dz}(z-i)^2 \cdot \frac{1}{(z-i)^2}\bigg|_{z=i} = \frac{d}{dz}1\bigg|_{z=i} = 0.$$

If $k > 0$, then the pole at $z = i$ is of order $k + 2$, and the residue is

$$\frac{d^{k+1}}{dz^{k+1}}(z-i)^{k+2} \cdot \frac{(z+i)^k}{(z-i)^{k+2}}\bigg|_{z=i} = \frac{d^{k+1}}{dz^{k+1}}(z+i)^k\bigg|_{z=i} = 0.$$

The last derivative is zero everywhere because $(z+i)^k$ is a polynomial of degree k.

Since the residue is zero for $k \geq 0$,

$$\oint_\Gamma \frac{(z+i)^k}{(z-i)^{k+2}}\; dz = 0.$$

It remains for us to show that the integral along the semicircle decays to zero as $R \to \infty$.

On the semicircle, $z = Re^{i\theta}$, $dz = iRe^{i\theta}\; d\theta$.

$$\left|\int_{\text{semicircle}} \frac{(z+i)^k}{(z-i)^{k+2}}\; dz\right| = \left|\int_0^\pi \frac{(Re^{i\theta}+i)^k}{(Re^{i\theta}-i)^{k+2}}\; iRe^{i\theta}\; d\theta\right| \leq \int_0^\pi \frac{(R+1)^k}{(R-1)^{k+2}}\; R\; d\theta$$

Perhaps this last inequality deserves some explanation. Consider each complex number as a vector in $\mathbb{R}^2$. The magnitude or modulus of $Re^{i\theta}+i$ is greatest when the vectors $Re^{i\theta}$ and i point in the same direction. When they point in the same direction, their magnitudes add, so

$$\left|Re^{i\theta}+i\right| \leq \max_\theta \left|Re^{i\theta}+i\right| = R+1.$$

Similarly, the magnitude of $Re^{i\theta} - i$ is smallest when the vectors $Re^{i\theta}$ and $-i$ point in opposite directions. When the vectors point in opposite directions, the magnitude of $Re^{i\theta}+i$ is equal to the magnitude of the larger minus the magnitude of the smaller. We are considering large values of R, so $R > 1$, and

$$\left|Re^{i\theta}-i\right| \geq \min_\theta \left|Re^{i\theta}-i\right| = R-1.$$

The magnitude of the quotient is bounded as follows.

$$\left|\frac{(Re^{i\theta}+i)^k}{(Re^{i\theta}-i)^{k+2}}\right| \leq \frac{(\max_\theta \left|Re^{i\theta}+i\right|)^k}{(\min_\theta \left|Re^{i\theta}-i\right|)^{k+2}} = \frac{(R+1)^k}{(R-1)^{k+2}}$$

The integral along the closed contour Γ consists of two parts:

$$\frac{1}{\pi}\int_{\text{semicircle}} \frac{(z+i)^k}{(z-i)^{k+2}}\; dz + \frac{1}{\pi}\int_{-R}^{R} \frac{(x+i)^k}{(x-i)^{k+2}}\; dx = 0.$$

Now we consider the $R \to \infty$ limit.

$$\begin{aligned}
\lim_{R\to\infty} \left| \frac{1}{\pi} \int_{-R}^{R} \frac{(x+i)^k}{(x-i)^{k+2}}\, dx \right| &= \lim_{R\to\infty} \left| \frac{1}{\pi} \int_{\text{semicircle}} \frac{(z+i)^k}{(z-i)^{k+2}}\, dz \right| \\
\left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x+i)^k}{(x-i)^{k+2}}\, dx \right| &\leq \lim_{R\to\infty} \frac{1}{\pi} \int_0^{\pi} \frac{(R+1)^k}{(R-1)^{k+2}}\, R\, d\theta \\
&= \lim_{R\to\infty} \frac{R(R+1)^k}{(R-1)^{k+2}} \\
&= \lim_{R\to\infty} \frac{R^{k+1}(1+\frac{1}{R})^k}{R^{k+2}(1-\frac{1}{R})^{k+2}} \\
&= \lim_{R\to\infty} \frac{(1+\frac{1}{R})^k}{R(1-\frac{1}{R})^{k+2}} \; = \; 0
\end{aligned}$$

The functions f_n, $n \in \mathbb{Z}$, form an orthonormal set with respect to the given inner product.

13 Day 2, Question 2

> Let ℓ^2 be the Hilbert space of square summable sequences. Let $\boldsymbol{x}^{(n)}$ be a sequence in ℓ^2 with $\|\boldsymbol{x}^{(n)}\|_2$ bounded. Let $\boldsymbol{x}^{(n)} = \left(x_1^{(n)}, x_1^{(n)}, \ldots\right)$. Suppose there is an $\boldsymbol{x} = (x_1, x_2, \ldots) \in \ell^2$ such that for every k, $\lim_{n\to\infty} x_k^{(n)} = x_k$. Prove that $\boldsymbol{x}^{(n)}$ converges weakly to $\boldsymbol{x}$. (Recall that $\boldsymbol{x}^{(n)}$ weakly to $\boldsymbol{x}$ if for every $\boldsymbol{y} \in \ell^2$ we have $\left(\boldsymbol{x}^{(n)}, \boldsymbol{y}\right) \to (\boldsymbol{x}, \boldsymbol{y})$.)

It is important that the norms $\|\boldsymbol{x}^{(n)}\|_2$ are bounded. Let M be the bound.

Fix $\boldsymbol{y} \in \ell^2$. Then

$$\left(\boldsymbol{x}^{(n)}, \boldsymbol{y}\right) - (\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{\infty} \left(x_k^{(n)} - x_k\right) y_k$$

Since $\boldsymbol{y} \in \ell^2$, there is some $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} y_k^2 < \frac{\epsilon^2}{4(M + \|\boldsymbol{x}\|_2)^2}.$$

Now we break the series into two pieces:

$$\sum_{k=1}^{\infty} \left(x_k^{(n)} - x_k\right) y_k = \sum_{k=1}^{N} \left(x_k^{(n)} - x_k\right) y_k + \sum_{k=N+1}^{\infty} \left(x_k^{(n)} - x_k\right) y_k.$$

We now prove that the second term is small. By the Cauchy-Schwarz Inequality,

$$\left| \sum_{k=N+1}^{\infty} \left(x_k^{(n)} - x_k\right) y_k \right| \leq \left[\sum_{k=N+1}^{\infty} \left(x_k^{(n)} - x_k\right)^2 \right]^{1/2} \left[\sum_{k=N+1}^{\infty} y_k^2 \right]^{1/2}.$$

By the Minkwoski Inequality,

$$\left[\sum_{k=N+1}^{\infty}\left(x_k^{(n)}-x_k\right)^2\right]^{1/2} \;\le\; \|\boldsymbol{x}^{(n)}-\boldsymbol{x}\|_2 \;\le\; \|\boldsymbol{x}^{(n)}\|_2+\|\boldsymbol{x}\|_2 \;\le\; M+\|\boldsymbol{x}\|_2.$$

By definition of N,

$$\left[\sum_{k=N+1}^{\infty}|y_k|^2\right]^{1/2} < \frac{\epsilon}{2(M+\|\boldsymbol{x}\|_2)},$$

so

$$\left|\sum_{k=N+1}^{\infty}\left(x_k^{(n)}-x_k\right)y_k\right| \;<\; 2(M+\|\boldsymbol{x}\|_2)\left(\frac{\epsilon}{2(M+\|\boldsymbol{x}\|_2)}\right) \;=\; \frac{\epsilon}{2}.$$

Now we consider the first portion of the series. Since $\boldsymbol{y}$ is fixed, there is a single bound on the terms $y_1,\ldots,y_N$. Let K be a bound on these terms. Since $x_1^{(n)}\to x_1,\ldots,x_N^{(n)}\to x_N$ as $n\to\infty$, there is some $\widetilde{N}\in\mathbb{N}$ such that

$$\left|x_k^{(n)}-x_k\right| < \frac{\epsilon}{2KN},\quad k=1,\ldots,N,\quad \text{for}\;\; n\ge\widetilde{N}.$$

The sum from $k=1$ to $k=N$ then satisfies

$$\begin{aligned}\left|\sum_{k=1}^{N}\left(x_k^{(n)}-x_k\right)y_k\right| \;&\le\; K\sum_{k=1}^{N}\left|x_k^{(n)}-x_k\right|\\ &<\; K\sum_{k=1}^{N}\frac{\epsilon}{2KN}\\ &=\; K\cdot N\cdot\frac{\epsilon}{2KN} \;=\; \frac{\epsilon}{2}.\end{aligned}$$

Hence for each $\boldsymbol{y}\in\ell^2$, for each $\epsilon>0$ there is some $\widetilde{N}\in\mathbb{N}$ such that for $n\ge\widetilde{N}$,

$$\left|\left(\boldsymbol{x}^{(n)},\boldsymbol{y}\right)-(\boldsymbol{x},\boldsymbol{y})\right| = \left|\sum_{k=1}^{\infty}\left(x_k^{(n)}-x_k\right)y_k\right| \;<\; \epsilon.$$

We conclude that $\left(\boldsymbol{x}^{(n)},\boldsymbol{y}\right)\to(\boldsymbol{x},\boldsymbol{y})$ as $n\to\infty$.

14 Day 2, Question 5

Let (X_1,d_1), (X_2,d_2) be metric spaces and $f: X_1\to X_2$ a continuous map *onto* X_2 such that

$$d_1(p,q) \;\le\; d_2(f(p),f(q))$$

for every $p,q\in X_1$.

(a) If X_1 is complete, must X_2 be complete?
(b) If X_2 is complete, must X_1 be complete?
In each case, give a proof or a counterexample.

Recall that a complete metric space includes all of its limit points. That is, if $\{x_n\}$ is a Cauchy sequence in metric space (X, d) and has limit x, then $x \in X$.

The answer to part (a) is affirmative. The intuition to have is the following. Since f is *onto* X_2, each $y \in X_2$ is the image under f of at least one $x \in X_1$. If $\{y_n\}$ is a Cauchy sequence in (X_2, d_2), then there is at least one sequence $\{x_n\}$ of pre-images of $y_1, y_2, \ldots$. Since

$$d_1(x_m, x_n) \;\le\; d_2(f(x_m), f(x_n)) \;=\; d_2(y_m, y_n),$$

the sequence $\{x_n\}$ is a Cauchy sequence in (X_1, d_1) and thus has a limit point x. Since (X_1, d_1) is a complete metric space, this x is in X_1. Since the domain of f is X_1 and since $x \in X_1$, there is a point $f(x) \in X_2$. Since f is continuous,

$$\lim_{n\to\infty} y_n \;=\; \lim_{n\to\infty} f(x_n) \;=\; f\left(\lim_{n\to\infty} x_n\right) \;=\; f(x).$$

We've already established that $f(x) \in X_2$, and now we know that this is the limit of the sequence $\{y_n\}$. Hence (X_2, d_2) contains the limit point of each cauchy sequence. (X_2, d_2) is complete.

A problem with this proof is the reference to sequences $\{x_n\}$. The pre-image $f^{-1}(y_n)$ might have more than one point of X_1 in it, and creating a sequence of *single* pre-image points requires the existence of a choice function, so we are relying on the Axiom of Choice.

The answer to part (b) is negative. Let

$$\begin{aligned} X_1 = (0,1], &\qquad d_1(x,y) = |x-y|\,, \\ X_2 = [1,\infty), &\qquad d_2(p,q) = |p-q|. \end{aligned}$$

Recall that $\mathbb{R}$ with the metric $d(x,y) = |x-y|$ is a complete metric space. Since X_2 is a closed subset of $\mathbb{R}$, (X_2, d_2) is also a complete metric space. (X_1, d_1) is not complete, as the number 1 can be the limit point of a Cauchy sequence in (X_1, d_1), but $1 \notin X_2$.

If $f(x) = x^{-1}$, then $f : X_1 \to X_2$ is onto X_2. Let $x, y \in (0, 1]$. Then

$$\begin{aligned} d_2(f(x), f(y)) &= |f(x) - f(y)| \\ &= \left|\frac{1}{x} - \frac{1}{y}\right| \\ &= \left|\frac{y-x}{xy}\right| \\ &\ge |y - x| \qquad (0 < x, y \le 1) \\ &= d_1(x,y). \end{aligned}$$

It still remains to show that f is continuous. Fix $x \in (0, 1]$. You should note that fixing x beforehand indicates that we are not seeking to show *uniform* continuity. Let $\delta > 0$ be small enough that $x \pm \delta \in (0, 1]$.

$$\begin{aligned}
\left|\frac{1}{x} - \frac{1}{x \pm \delta}\right| &= \left|\frac{x \pm \delta - x}{x(x \pm \delta)}\right| \\
&= \frac{\delta}{x(x \pm \delta)} \\
&\leq \frac{\delta}{x(x - \delta)}
\end{aligned}$$

Fix $\epsilon > 0$. We seek a value of δ that will make the difference above less than ϵ.

$$\begin{aligned}
\frac{\delta}{x(x - \delta)} &< \epsilon \\
\delta &< \epsilon x(x - \delta) \\
(1 + \epsilon x)\delta &< \epsilon x^2 \\
\delta &< \frac{\epsilon x^2}{1 + \epsilon x}
\end{aligned}$$

If δ is this small and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. f is continuous but not uniformly continuous.

15 Day 2, Question 7

Let the real-valued functions $f_1, \ldots, f_{n+1}$ on $\mathbb{R}$ satisfy the system of differetial equations

$$\begin{aligned}
f'_{k+1} + f'_k &= (k+1)f_{k+1} - kf_k, \qquad 1 \leq k \leq n, \\
f'_{n+1} &= -(n+1)f_{n+1}.
\end{aligned}$$

Prove that for each k, $\lim_{t\to\infty} f_k(t) = 0$.

Consider f_{n+1} first. It's clear that

$$f_{n+1}(t) = c_{n+1}e^{-(n+1)t}$$

for some constant c_{n+1}. Now consider the equation with $k = n$.

$$\begin{aligned}
f'_{n+1} + f'_n &= (n+1)f_{n+1} - nf_n \\
-(n+1)c_{n+1}e^{-(n+1)t} + f'_n &= (n+1)c_{n+1}e^{-(n+1)t} - nf_n
\end{aligned}$$

In order to get an equality, we try $f_n(t) = c_n e^{-(n+1)t}$.

$$\begin{aligned}
-(n+1)c_{n+1}e^{-(n+1)t} - (n+1)c_n e^{-(n+1)t} &= (n+1)c_{n+1}e^{-(n+1)t} - nc_n e^{-(n+1)t} \\
(n - (n+1))c_n &= 2(n+1)c_{n+1} \\
c_n &= -2(n+1)c_{n+1}
\end{aligned}$$

Now we consider induction. Assume that $f_{k+1} = c_{k+1}e^{-(n+1)t}$, which we know works for $k = n$. We want to show that $f_k = c_k e^{-(n+1)t}$.

$$\begin{aligned} f'_{k+1} + f'_k &= (k+1)f_{k+1} - kf_k \\ -(n+1)c_{k+1}e^{-(n+1)t} + f'_k &= (k+1)c_{k+1}e^{-(n+1)t} - kf_k \end{aligned}$$

We consider $f_k = c_k e^{-(n+1)t}$ to get equality and find

$$\begin{aligned} -(n+1)c_{k+1}e^{-(n+1)t} - (n+1)c_k e^{-(n+1)t} &= (k+1)c_{k+1}e^{-(n+1)t} - kc_k e^{-(n+1)t} \\ (k-(n+1))c_k &= (n+k+2)c_{k+1} \end{aligned}$$

This gives a formula for c_k. This can be iterated to produce $c_n, c_{n-1}, \ldots, c_2, c_1$ that make equalities out of the differetial equations above.

Since this is a system of linear first-order equations with constant coefficients, once initial conditions are specified, the solution is unique. Since the solution has the form $f_k = c_k e^{-(n+1)t}$, $k = 1, \ldots, n+1$, $f_k(t) \to 0$ as $t \to \infty$.

16 Day 2, Question 8

Does there exist a 2×2 matrix $\boldsymbol{A}$ with complex entries that has no square root, *i.e.*, there is no 2×2 matrix $\boldsymbol{B}$ such that $\boldsymbol{B}^2 = \boldsymbol{A}$?

First note that if $\boldsymbol{A}$ is diagonalizable, then $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \boldsymbol{C} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \boldsymbol{C}^{-1},$$

where λ_1, λ_2 are the eiganvalues of $\boldsymbol{A}$. If we define $\boldsymbol{B}$ by

$$\boldsymbol{B} = \boldsymbol{C} \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} \boldsymbol{C}^{-1},$$

where any square roots can be chosen, then

$$\begin{aligned} \boldsymbol{B}^2 &= \boldsymbol{C} \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} \boldsymbol{C}^{-1}\boldsymbol{C} \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} \boldsymbol{C}^{-1} \\ &= \boldsymbol{C} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \boldsymbol{C}^{-1} \\ &= \boldsymbol{A}. \end{aligned}$$

Hence, if there is a 2×2 matrix with no square root, it must not be diagonalizable. Let's try the simplest non-diagonalizable 2×2 matrix we can construct: a 2×2 "nontrivial" Jordan block:

$$\boldsymbol{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

$\boldsymbol{A}$ has eigenvalue λ with *algebraic multiplicity* 2 but *geometric multiplicity* 1.

Suppose that there is a 2×2 matrix $\boldsymbol{B}$ such that $\boldsymbol{B}^2 = \boldsymbol{A}$. Suppose that $\boldsymbol{B}$ has two linearly independent eigenvectors, perhaps with the same eigenvalue. Since $\boldsymbol{B}$ has two linearly independent eigenvectors, $\boldsymbol{B}^2$ also has two linearly independent eigenvectors. But $\boldsymbol{A}$ has just one eigenvector, so $\boldsymbol{B}$ cannot have two linearly independent eigenvectors.

There is only one other option: suppose that $\boldsymbol{B}$ has just one eigenvector. Then the Jordan form of $\boldsymbol{B}$ is also a nontrivial 2×2 Jordan block:

$$\boldsymbol{B} = \boldsymbol{C} \begin{pmatrix} \sqrt{\lambda} & 1 \\ 0 & \sqrt{\lambda} \end{pmatrix} \boldsymbol{C}^{-1},$$

where $\sqrt{\lambda}$ can be any square root of λ, but the same square root must be put in the $(1,1)$ and $(2,2)$ positions. If $\boldsymbol{B}$ has this form, then $\boldsymbol{B}^2$ has the form

$$\begin{aligned} \boldsymbol{B}^2 &= \boldsymbol{C} \begin{pmatrix} \sqrt{\lambda} & 1 \\ 0 & \sqrt{\lambda} \end{pmatrix} \boldsymbol{C}^{-1} \boldsymbol{C} \begin{pmatrix} \sqrt{\lambda} & 1 \\ 0 & \sqrt{\lambda} \end{pmatrix} \boldsymbol{C}^{-1} \\ &= \boldsymbol{C} \begin{pmatrix} \sqrt{\lambda} & 1 \\ 0 & \sqrt{\lambda} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda} & 1 \\ 0 & \sqrt{\lambda} \end{pmatrix} \boldsymbol{C}^{-1} \\ &= \boldsymbol{C} \begin{pmatrix} \lambda & 2\sqrt{\lambda} \\ 0 & \lambda \end{pmatrix} \boldsymbol{C}^{-1}. \end{aligned}$$

If we choose $\boldsymbol{C} = \boldsymbol{I}$, then

$$\boldsymbol{B} = \begin{pmatrix} \sqrt{\lambda} & 1 \\ 0 & \sqrt{\lambda} \end{pmatrix} \quad \text{and} \quad \boldsymbol{B}^2 = \begin{pmatrix} \lambda & 2\sqrt{\lambda} \\ 0 & \lambda \end{pmatrix},$$

the latter of which matches our choice of $\boldsymbol{A}$ if and only if $2\sqrt{\lambda} = 1$. If $\lambda \neq 1/4$, then

$$\boldsymbol{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

has no square root.

The example that will be easiest to remember is

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This has two zero eigenvalues, so any square root would also have to have two zero eigenvalues. Since a 2×2 matrix $\boldsymbol{B}$ with two zero eigenvalues and two linearly independent eigenvectors would be the zero matrix, we assume that $\boldsymbol{B}$ has just one eigenvector. Such a $\boldsymbol{B}$ has the Jordan form

$$\boldsymbol{J} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \boldsymbol{J}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The latter matrix is not similar to the chosen $\boldsymbol{A}$, so this $\boldsymbol{A}$ has no square root.